Question 3

Show that $Cov(X) = E(XX') - \mu\mu'$
$\mu = E(X)$

$Var(X) = E((X-\mu_X)^2) = Cov(X,X)$.

$Var(X) = E(X^2) - \mu^2_X$.

$Cov(X) = E((X-\mu_X)(X'-\mu_{X'}))$

$= E(XX' - \mu_X X' - X\mu_{X'} + \mu_X \mu_{X'})$

$= E(XX') - \mu_X E(X') - E(X)\mu_{X'} + \mu_X \mu_{X'}$

$= E(XX') - \mu\mu'$

Question 4

Show that $(i,j)^{th}$ term of $H = X(X'X)^{-1}X'$

is $h_{ij} = \frac{1}{n} + \frac{(x_i - \bar{x})(x_j - \bar{x})}{S_{xx}}$

$$Y = X\beta + \epsilon$$

$$X'X\beta = X'Y$$

$$\hat{\beta} = (X'X)^{-1}X'Y$$

$$\hat{Y} = X\hat{\beta} = X(X'X)^{-1}X'Y$$

$$var(\hat{Y}_i) = \sigma^2 \sum_{j=1}^{n} h_{ij}^2$$

$$= \sigma^2 \sum_{i=1}^{n} \left(\frac{1}{n} + \frac{(x_i - \bar{x})(x_j - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \right)^2$$

$$\frac{\sqrt{\sigma_i^2}}{\sigma^2} = \frac{\sigma^2}{\sigma^2} \sqrt{\sum_{j=1}^{n} \left(\frac{1}{n} + \frac{(x_i - \bar{x})(x_j - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \right)^2} \qquad S_{xx} = \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$= \frac{1}{n} + \frac{(x_i - \bar{x})(x_j - \bar{x})}{S_{xx}}$$

Show that the trace of H is $=2$

If X is $i \times j$ with $j \leq i$ and has a full rank, then $rank(X) = min\{i,j\} = j$ and we know that $(X'X)^{-1}$ exists

By commutativity of the trace operator

$$tr(H) = tr(X(X'X)^{-1}X')$$

$$= tr(X'X(X'X)^{-1}) = tr(I_m) = m$$

but H is 2×2 matrix hence $m = 2$

$$tr\{I_2\} = 2$$

Question 5 (a)

$Y = AX$ $\quad X = (X_1, X_2)'$ $\quad A$ is 2×2 matrix of constants

(a) Derive the elements of the vector $Y = (Y_1, Y_2)'$ by matrix multiplication.

$$CoV(Y) = Cov(Y_1, Y_2)$$

$$\equiv \begin{pmatrix} Var(Y_1) & Cov(Y_1 Y_2) \\ Cov(X_2 Y_1) & Var(Y_2) \end{pmatrix}$$

$$Y = A \cdot X$$

$$\begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \begin{pmatrix} X_1 \cdots X_n \\ X_2 \cdots X_2 \\ \vdots \\ X_n \cdots X_{n+2} \end{pmatrix}$$

$$= \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{matrix} a_1 X_1 + X_2 a_2 \\ a_2 X_1 + a_2 X_2 \end{matrix}$$

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} a_1 X_1 & a_1 X_2 \\ a_2 X_1 & a_2 X_2 \end{pmatrix}$$

Question 5b

$$Cov(Y) = \begin{pmatrix} Var\, Y_1 & Cov(Y_1, Y_2) \\ Cov(Y_1, Y_2 & Var(Y_2) \end{pmatrix}$$

$$VAR[Y] = E[YY'] - E(Y) \cdot E(\cancel{Y}^{-1})$$

Let Y is a 2×1 random vector with components Y_1 & Y_2.)

Let $Var(Y_1) = 2$

$Var(Y_2) = 4$

$Cov(Y_1, Y_2) = 1$

By the symmetry of covariance it must also be

$$Cov(Y_1 Y_2) = Cov(Y_2 Y_1) = 1$$

$$Cov(Y) = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

Question 5 (C)

Show by matrix multiplication

$Cov(Y) = A\, Cov(X)\, A'$

$Cov(Y) = A\, Cov(X)\, A'$

Proof.

$$Cov(Y) = E[(AY - E[AY])(AY - E[AY])']$$

$$= E[(AY - E[AY])(AY - E\,AY)']$$

$$= A\,E[(X - E[X])(X - E[X])']\,A'$$

$$= A\,Cov(X)\,A'$$

Question 7 (a)

$X_1, \dots, X_n$ are iid g.r.v with parameter p.

$f(x) = (1-p)^{x-1} p$ for $x = 1, 2, \dots$

Find maximum likelihood estimator for p.

$$L(p) = \prod_{i=1}^{n} p(1-p)^{x-1}$$

$$L(p) = p^n (1-p)^{\Sigma x - n}$$

Taking ln both sides

$$\ln L(p) = n \ln p + \Sigma(x-n) \ln(1-p)$$

differentiate with respect to p

$$\frac{d}{dp} \ln L(p) = n \frac{d}{dp} \ln p + (\Sigma x - n) \frac{d}{dp} \ln(1-p).$$

$$\frac{d}{dp} \ln L(p) = \frac{n}{p} - \frac{1}{1-p}(\Sigma x - n) \quad \text{--- (i)}$$

put $\frac{d}{dp} \ln L(p) = 0$

$$\frac{n}{p} - \frac{1}{1-p}(\Sigma x - n) = 0$$

$$n(1-p) = p(\Sigma x - n)$$

$$p = \frac{n}{\Sigma x}$$

(7a)

find 2nd derivative with respect to p

$$\frac{d^2}{dp^2} \ln L(p) = \frac{-n}{p^2} - \frac{(\Sigma x - n)}{(1-p)^2}$$

$$\frac{d^2}{dp^2} \ln L(p) = \frac{-n(1-p)^2 - p^2(\Sigma x - n)}{p^2(1-p)^2}$$

$$= \frac{-n + np}{p^2(1-p)^2}$$

$$= \frac{-n}{p^2(1-p)} < 0 \text{ as } 0 < p < 1$$

So Maximum Likelihood Estimator of p

is $\hat{p} = \frac{n}{\Sigma x} = \frac{1}{\frac{\Sigma x}{n}} = \frac{1}{\bar{x}}$

$$= \frac{1}{\bar{x}}$$

$$= \frac{-n+np}{p^2(1-p)^2}$$

$$= \frac{-n}{p^2(1-p)} < 0 \text{ as } 0 < p < 1$$

So Maximum Likelihood Estimator of p

i.e. $\hat{p} = \frac{n}{\sum x} = \frac{1}{\frac{\sum x}{n}} = \frac{1}{\bar{x}}$

$$= \frac{1}{\bar{x}}$$

7 (b) Show that this is not equal to the moment estimator of p.

Moment Generating function.

$$M(t) = E(e^{tx})$$

$$= \sum_{x=1}^{\infty} e^{tx}(1-p)^{x-1}p$$

$$= p\sum_{x=1}^{\infty} e^{tx}e^{t}e^{-t}(1-p)^{x-1}$$

$$= pe^{t}\sum_{x=1}^{\infty} e^{t(x-1)}(1-p)^{x-1}$$

Scanned with CamScanner

(7b)

$$= p\, e^t (1 + \theta + \theta^2 \ldots$$

$$= \frac{p\, e^t}{1-\theta} \quad \text{if } |\theta| < 1$$

$$= \frac{p e^t}{1 - q e^t}$$

$$= \frac{1}{p}$$

hence $\frac{1}{x} \neq \frac{1}{p}$